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# SOLVING THE EQUATIONS OF UNIFORM FLOW


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## ABSTRACT


The problem of solving the equation for uniform flow in an open channel is reduced to that of evaluating three simple integrals. Practical methods of computing these integrals are discussed.


(A)

# SOLVING THE EQUATIONS OF UNIFORM FLOW*

## 1. Introduction to the Problem

The equation for uniform flow in an open channel is**

$$\frac{dy}{dx} = \frac{S_o - S_f}{1 - Z_c^2/Z^2} \qquad [1]$$

where $y$ is the stream depth, $x$ is the distance along the channel bed, $Z_c$ is the section factor at critical flow and $Z$ is the section factor of the actual flow, and $S_o$ is the slope of the stream bed; $S_f$ is the energy grade line and is given by

$$S_f = Q^2/K^2 \qquad [2]$$

where $Q$ is the flowrate and $K$ is the conveyance.

The section factor is given by (Ref. 1, Ch. 4)

$$Z^2 = C_z y^M \qquad [3]$$

where $C_z$ and $M$ are positive constants.

The conveyance is given by (Ref. 1, Ch. 6)

$$K^2 = C_k y^N \qquad [4]$$

where $C_k$ and $N$ are positive constants.

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**The reference for this equation and for the definitions and properties of the section factor and the conveyance is the treatise of Chow (1) on open channel hydraulics. The reference also gives the reader an excellent introduction to the physics of the problem and discusses previously devised methods of solving it.

[2], [3], and [4] can be substituted into [1] to yield

$$dx = \frac{1 - (y_c/y)^M}{S_o - (Q^2/C_k)y^{-N}} dy \qquad [5]$$

where the critical depth $y_c$ is given by

$$y_c = (Z_c{}^2/C_z)^{1/M} \quad . \qquad [5']$$

In most applications [5] is integrated to give the distance $\delta$ between two depths $y_1$ and $y_2$:

$$\delta = \int_{y_1}^{y_2} \frac{1 - (y_c/y)^M}{S_o - (Q^2/C_k)y^{-N}} dy \quad y_2 \geq y_1 \quad . \qquad [6]$$

The only mathematical consideration to be observed in taking the integral of [6] is that no zero of the denominator of the integrand be permitted to occur over the range $(y_1, y_2)$*. [6] can be given the more convenient form

$$\delta = B(M, N; S_o, y_c, Q^2/C_k; y_2) - B(M, N; S_o, y_c, Q^2/C_k; y_1) \qquad [7]$$

where

$$B(M, N; S_o, y_c, Q^2/C_k; y) = \int \frac{1 - (y_c/y)^M}{S_o - (Q^2/C_k)y^{-N}} dy \quad . \qquad [7']$$

It is convenient to study the function B .

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*Physically, this would correspond to prohibiting the flow profile in a uniform channel from smoothly crossing the normal depth. This is in accord with experimental fact.

## 2. The Reduction of $B(M, N; S_o, y_c, Q^2/C_k; y)$ to Its Four Normal Forms

To evaluate the B-function it is convenient to divide the fifteen hydraulic profiles into five classes.

Class I: M2, M3, S3, C3.

In this class $S_o > 0$ and $\begin{Bmatrix} y_n > y_1 \\ y_n > y_2 \end{Bmatrix}$ where the normal depth $y_n$ is given by

$$y_n = \left( \frac{Q^2}{C_k S_o} \right)^{1/N} . \qquad [8]$$

The substitution of [8] into [7'] yields

$$B_I(y) = \frac{1}{S_o} \int \frac{1 - (y_c/y_n)^M (y_n/y)^M}{1 - (y_n/y)^N} \, dy \qquad [9]$$

$$0 < y < y_n < \infty$$

which, subject to the substitution

$$\lambda_1 = (y_n/y)^{-N} \qquad [10]$$

becomes

$$B_I(y) = - \frac{y_n}{N S_o} \int \frac{\lambda_1^{1/N} - (y_c/y_n)^M \lambda_1^{\frac{1-M}{N}}}{1 - \lambda_1} \, d\lambda_1 . \qquad [11]$$

Class II: M1, S1, S2, C1.

In this class $S_o > 0$ and $\begin{Bmatrix} y_n < y_1 \\ y_n < y_2 \end{Bmatrix}$ where $y_n$ is given by [8].

The substitution

$$\lambda_2 = (y_n/y)^N \qquad [12]$$

reduces [9] to

$$B_{II}(y) = -\frac{y_n}{NS_o} \int \frac{\lambda_2^{-\frac{1+N}{N}} - (y_c/y_n)^M \lambda_2^{\frac{M-1-N}{N}}}{1-\lambda_2} \, d\lambda_2 \qquad [13]$$

$$\infty > y > y_n > 0 \, .$$

Class III: H2, H3.

In this class $S_o = 0$. Thus [7'] reduces to

$$B_{III}(y) = \frac{-1}{(Q^2/C_k)} \int \left[ y^N - y_c^M \; y^{N-M} \right] dy \quad 0 < y < \infty \quad . \qquad [14]$$

Class IV: A2, A3.

In this class $S_o < 0$ and [7'] reduces to

$$B_{IV}(y) = \frac{1}{S_o} \int \frac{1 - (y_c/y_a)^M (y_a/y)^M}{1 + (y_a/y)^N} \, dy \qquad [15]$$

where the adverse depth $y_a$ is given by

$$y_a = \left| \frac{Q^2}{S_o C_k} \right|^{1/N} \quad . \qquad [15']$$

Suppose $y_a/y \geq 1$ . Define

$$\lambda_3 = (y_a/y)^{-N} \quad . \qquad [16]$$

Then

$$B_{IV}(y) = \frac{y_a}{NS_o} \int^{(y_a/y)^{-N}} \frac{\lambda_3^{1/N} - (y_c/y_a)^M \lambda_3^{\frac{1-M}{N}}}{1+\lambda_3} \, d\lambda_3 \quad . \qquad [17]$$

$$0 < y \lesssim y_a < \infty$$

Suppose $y_a/y < 1$. Define

$$\lambda_4 = (y_a/y)^N \quad . \qquad [18]$$

Then

$$B_{IV}(y) = B_{IV}(y_a) - \frac{y_a}{NS_o}\int_1^{(y_a/y)^N} \frac{\lambda_4^{-\frac{1-N}{N}} - (y_a/y_a)^M \lambda_4^{\frac{M-1-N}{N}}}{1+\lambda_4} d\lambda_4 \qquad [19]$$

$$y_a \leqslant y < \infty \quad .$$

Class V: H1, A1; C2.

H1 and A1 are not physically realizable (Ref. 1, Ch. 9). C2 is the trivial case $y_c = y = y_n$ ; it is unstable.

The most interesting fact about the integrals thus far presented is that they can be simply expressed in terms of three fundamental functions, the backwater integrals:

$$\mathcal{B}_\gamma^{(1)}(z) = \int \frac{z^{\gamma - 1}}{1 - z} dz \qquad 0 < z < 1 \qquad [20a]$$

$$\mathcal{B}_\gamma^{(2)}(z) = \int z^{\gamma - 1} dz \qquad 0 < z < \infty \qquad [20b]$$

$$\mathcal{B}_\gamma^{(3)}(z) = \int \frac{z^{\gamma - 1}}{1 + z} dz \qquad 0 < z < 1 \quad . \qquad [20c]$$

One can thus reduce the B's to

$$B_I(y) = -\frac{y_n}{NS_o}\left[\mathcal{B}^{(1)}_{\frac{1+N}{N}}(a_1) - (y_c/y_n)^M \mathcal{B}^{(1)}_{\frac{1-M+N}{N}}(a_1)\right] \qquad [21]$$

$$a_1 = (y_n/y)^{-N} \qquad [21']$$

$$B_{II}(y) = -\frac{y_n}{NS_o}\left[\mathcal{B}^{(1)}_{-1/N}(\alpha_2) - (y_c/y_n)^M \mathcal{B}^{(1)}_{\frac{M-1}{N}}(\alpha_2)\right] \quad [22]$$

$$\alpha_2 = (y_n/y)^N \quad [22']$$

$$B_{III}(y) = \frac{-1}{(Q^2/C_k)}\left[\mathcal{B}^{(2)}_{N+1}(\alpha_3) - y_c^M \; \mathcal{B}^{(2)}_{N-M+1}(\alpha_3)\right] \quad [23]$$

$$\alpha_3 = y \quad [23']$$

$$y_a \leq y \quad B_{IV}(y) = \frac{y_a}{NS_o}\left[\mathcal{B}^{(3)}_{\frac{1+N}{N}}(\alpha_4) - (y_c/y_a)^M \mathcal{B}^{(3)}_{\frac{1-M+N}{N}}(\alpha_4)\right] \quad [24]$$

$$\alpha_4 = (y_a/y)^{-N} \quad [24']$$

$$y_a \geq y \qquad = B_{IV}(y_a) - \frac{y_a}{NS_o}\left[\mathcal{B}^{(3)}_{-\frac{1}{N}}(\lambda_4) - (y_a/y_a)^M \mathcal{B}^{(3)}_{\frac{M-1}{N}}(\lambda_4)\right] . \quad [24'']$$

In the next section methods of evaluating the $\mathcal{B}^{(l)}_{\gamma}(z)$ will be discussed.

## 3. Evaluation of the Backwater Integrals

The evaluation of the backwater integrals for all values of $\gamma$ presents no intrinsic difficulty. However, it can become rather involved, especially for $\gamma = -m (m = 0, 1, 2, \ldots)$, and presents practical problems in computation. For the purposes of this study it is sufficient, for $\mathcal{B}^{(1)}_{\gamma}$ and $\mathcal{B}^{(3)}_{\gamma}$, to consider only those portions of the $\gamma$-axis covered by $\frac{1+N}{N}$, $\frac{1-M+N}{N}$, $-\frac{1}{N}$, and $\frac{M-1}{N}$; or, for $\mathcal{B}^{(2)}_{\gamma}$, by $1+N$ and $1-M+N$. One can first note that for an open channel it generally happens that (1, p. 132) $2 \leqslant N < 6$; hence

$$-\frac{1}{2} \leqslant -\frac{1}{N} < -1/6 \qquad [25a]$$

$$3/2 \geqslant \frac{1+N}{N} > 7/6 \quad . \qquad [25b]$$

Similarly, using the values of $M$ calculated by Chow (1, p. 67), one has

$$1 > \frac{M-1}{N} > 1/2 \qquad [25c]$$

$$0 < \frac{1-M+N}{N} < 1/2 \quad . \qquad [25d]$$

Thus, for the evaluation of $\mathcal{B}_{\gamma}^{(1)}$ and $\mathcal{B}_{\gamma}^{(3)}$, the ranges of $\gamma$ which are of interest are

$$-\frac{1}{2} \leqslant \gamma < 0 \qquad [26a]$$

$$0 < \gamma \leqslant 3/2 \quad . \qquad [26b]$$

For the evaluation of $\mathcal{B}_{\gamma}^{(2)}$ the $\gamma$-range of interest is

$$0 < \gamma < 7 \quad . \qquad [27]$$

The restrictions placed upon $\gamma$ by [26] and [27] preclude the existence of logarithmic terms in any of the $\mathcal{B}_{\gamma}^{(\ell)}(z)$. Hence, it is convenient to expand the integrands of [20a] and [20c] in infinite series and, uniform continuity being seen to obtain, to integrate these series term by term:

$$\mathcal{B}_{\gamma}^{\binom{1}{3}}(z) = \sum_{n=0}^{\infty} (\pm 1)^n \frac{1}{n+\gamma} z^{n+\gamma} \quad . \qquad [28]$$

It happens that the series in [28] can be summed at once (2) to yield

$$\mathfrak{G}_{\gamma}^{(\frac{1}{3})}(z) = \frac{z^{\gamma}}{\gamma} \; F\,(1,\; \gamma;\; \gamma+1;\; \pm\, z) \qquad [29]$$

where F is an ordinary hypergeometric function. This fact, however, is of negligible value computationally and it is better to sum [28] by less direct methods.

[28] can be reduced at once to

$$\mathfrak{G}_{\gamma}^{(\frac{1}{3})}(z) = z^{\gamma}\left[\frac{1}{\gamma} + S_{\gamma}\,(\pm\, z)\right] \qquad [30]$$

where

$$S_{\gamma}\,(\pm\, z) = \sum_{n=1}^{\infty} \frac{(\pm\, 1)^{n}}{n+\gamma}\, z^{n} \qquad . \qquad [30']$$

By utilizing the relation (3)

$$\mathrm{Li}_{p}\,(\pm\, z) = \sum_{n=1}^{\infty} \frac{(\pm 1)^{n}}{n^{p}}\, z^{n} \qquad [31]$$

where $\mathrm{Li}_p$ is thepolyloga rithm function (3) of order p, it is possible to express [30'] as

$$S_{\gamma}\,(\pm\, z) = \sum_{k=1}^{K} (-\gamma)^{k-1}\, \mathrm{Li}_{k}\,(\pm\, z) + (-\gamma)^{K} \sum_{n=1}^{\infty} \frac{(\pm\, 1)^{n}}{n^{K}(n+\gamma)}\, z^{n} \qquad . \qquad [32]$$

The Eq. [32] can itself be simplified by noting that

$$\sum_{n=1}^{\infty} \frac{(\pm 1)^n}{n^K(n+\gamma)} z^n = z^{-\gamma} \int_0^z \sum_{n=1}^{\infty} \frac{(\pm 1)^n}{n^K} (z')^{n+\gamma-1}\, dz'$$

$$= z^{-\gamma} \int_0^z (z')^{\gamma-1}\, \mathrm{Li}_K(\pm z')\, dz' \quad . \qquad [33]$$

Let a polynomial approximation to $\mathrm{Li}_K$ be formed:

$$\mathrm{Li}_K(\pm z) = \sum_{j=1}^{J} {}^{J}_{K}c_j^{(\pm)} z^j + {}_K R_J^{(\pm)}(z) \quad . \qquad [34]$$

Then

$$S_\gamma(\pm z) = \sum_{j=1}^{J} {}^{J}_{0}c_j^{(\pm)} \frac{1}{j+\gamma} z^j + z^{-\gamma} \int_0^z (z')^{\gamma-1}\, {}_0R_J^{(\pm)}(z')\, dz' \quad K = 0 \qquad [35a]$$

$$= \sum_{k=1}^{K-1} (-\gamma)^{k-1}\, \mathrm{Li}_k(\pm z) + (-\gamma)^{K-1} \sum_{j=1}^{J} {}^{J}_{K}c_j^{(\pm)} \frac{j}{j+\gamma} z^j$$

$$+ (-\gamma)^{K-1}\, {}_K R_J^{(\pm)}(z) + (-\gamma)^K z^{-\gamma} \int_0^z (z')^{\gamma-1}\, {}_K R_J^{(\pm)}(z')\, dz' \quad K \geq 1 . \qquad [35b]$$

The choice of suitable K and J is not simple, since a balance must be drawn between the difficulties of computing the polylogarithm functions as opposed to those of summing series. In addition, $S_\gamma(+z)$ has a singularity at $z = 1$ which adversely affects the convergence of the series which obtain for $K \geq 0$ and makes it difficult to obtain suitable approximating polynomials.

To insure that the integrals in [33] were well behaved it was decided to approximate $\frac{Li_k(\pm z)}{(\pm z)}$ and then to represent ${}_K R_J^{(\pm)}(z')$ by

$$ {}_K R_J^{(\pm)}(z') = z' \; {}_K E_J^{(\pm)}(z') \tag{36}$$

where ${}_K E_J^{(\pm)}(z)$ is the error in $\frac{Li_K(\pm z)}{(\pm z)}$. If then, the maximum value of $|{}_K E_J^{(\pm)}(z)|$ over $(0, z)$ is ${}_K M_J^{(\pm)}(z)$ the error terms are, in absolute value, less than

$$\frac{z}{\gamma + 1} \; {}_o M_J^{(\pm)}(z) \tag{37a}$$

for [35a] and less than

$$\frac{|\gamma|^{K-1}}{\gamma + 1} \; z \, {}_K M_J^{(\pm)}(z) \tag{37b}$$

for [35b] .

The construction of the polynomial approximations is discussed in detail in Appendix I; it is sufficient to note the results here. For integrals of the third kind $K = 1$ was found to be adequate:

$${}_1 M_9^{(-)} < 5 \times 10^{-7} \tag{38.0}$$

$${}^9_1 c_1^{(-)} = -1.0000 \;\; 0000 \tag{38.1}$$

$${}^9_1 c_2^{(-)} = 0.4999 \;\; 8946 \tag{38.2}$$

$${}^9_1 c_3^{(-)} = -0.3330 \;\; 7718 \tag{38.3}$$

$${}^9_1 c_4^{(-)} = 0.2476 \;\; 3567 \tag{38.4}$$

$${}^9_1 c_5^{(-)} = -0.1883 \;\; 5208 \tag{38.5}$$

$${}^{9}_{1}c_6^{(-)} = 0.1315\ 6190 \tag{38.6}$$

$${}^{9}_{1}c_7^{(-)} = -0.0729\ 0260 \tag{38.7}$$

$${}^{9}_{1}c_8^{(-)} = 0.0265\ 6504 \tag{38.8}$$

$${}^{9}_{1}c_9^{(-)} = -0.0045\ 6742 \tag{38.9}$$

Thus,

$$\mathcal{B}_\gamma^{(3)}(z) = z^\gamma \left[\frac{1}{\gamma} + \sum_{j=1}^{9} {}^{9}_{1}c_j^{(-)} \frac{j}{j+\gamma} z^j + \mathcal{O}\left(5\times 10^{-7} \times \frac{z}{\gamma+1}\right)\right]. \tag{39}$$

For the integrals of the second kind

$$\mathcal{B}_\gamma^{(2)}(z) = \frac{1}{\gamma} z^\gamma \quad . \tag{40}$$

For integrals of the first kind, unfortunately, it proved necessary to go to K = 4 to obtain a polynomial approximation which converged with sufficient rapidity:

$${}_4M_9^{(+)} < 7\times 10^{-7} \times (1 - \ln(1 - z)) \quad . \tag{41.0}$$

(This odd weighting was chosen so that the relative accuracy of the approximation would not be greater than necessary near the singularity of $Li_1(z) = \ln(1-z)$ at $z = 1$.)

$${}^{9}_{4}c_1^{(+)} = 1.0000\ 0046 \tag{41.1}$$

$${}^{9}_{4}c_2^{(+)} = 0.0624\ 1793 \tag{41.2}$$

$${}^{9}_{4}c_3^{(+)} = 0.0143\ 7767 \tag{41.3}$$

$${}^{9}_{4}c_4^{(+)} = -0.0150\ 2213 \tag{41.4}$$

$$ {}^{9}_{4}c^{(+)}_{5} = 0.0873\ 5126 \tag{41.5}$$

$$ {}^{9}_{4}c^{(+)}_{6} = -0.2087\ 8801 \tag{41.6}$$

$$ {}^{9}_{4}c^{(+)}_{7} = 0.2831\ 2643 \tag{41.7}$$

$$ {}^{9}_{4}c^{(+)}_{8} = -0.1984\ 0826 \tag{41.8}$$

$$ {}^{9}_{4}c^{(+)}_{9} = 0.0572\ 7062 \tag{41.9}$$

Hence,

$$\mathcal{B}^{(1)}_{\gamma}(z) = z^{\gamma}\left[\frac{1}{\gamma} + \mathrm{Li}_1(z) - \gamma \mathrm{Li}_2(z) + \gamma^2 \mathrm{Li}_3(z) - \gamma^3 \sum_{j=1}^{9} {}^{9}_{4}c^{(+)}_{j} \frac{j}{j+\gamma} z^j + \mathcal{O}\left(7 \times 10^{-7} \times \frac{\gamma^3}{\gamma+1} \times z \times (1 - \ln(1-z))\right)\right] \tag{42}$$

Since the polylogarithms have been extensively tabulated(4), the forms [39], [40], and [42] provide a convenient way of computing the $\mathcal{B}^{\ell}_{\gamma}(z)$ by hand. The attainable accuracy will, of course, vary with the choice of $\ell$, $\gamma$, and z: it is believed that if eight figures are carried, the relative error in the answer will normally be less than 1 part in $10^5$.

The Eqs. [39] and [40] are well suited to use with digital computers. [42] is not quite as convenient for machine use since it requires the computation of three polylogarithm functions which can in turn necessitate the computation of five logarithms plus the summation of two slowly convergent series: it is advisable to compute $\mathfrak{G}_{\gamma}^{(3)}(z)$ directly from [30] when the series for $S_\gamma(z)$ converges with suitable rapidity. From [30']

$$S_\gamma(z) = \sum_{n=1}^{\infty} \frac{z^n}{n+\gamma} \qquad \begin{matrix} |z| < 1 \\ -1/2 < \gamma \leqslant 3/2 \end{matrix} \quad . \qquad [43]$$

To determine the z-range over which [43] is useful it is convenient to rewrite it as

$$S_\gamma(z) = \sum_{n=1}^{N} \frac{z^n}{n+\gamma} + T_N(z) \qquad [44]$$

where the truncation error $T_N(z)$ is given by

$$T_N(z) = \sum_{n=N+1}^{\infty} \frac{z^n}{n+\gamma} \quad . \qquad [44']$$

An application of Cauchy's integral method to[44']yields

$$|T_N(z)| \leqslant |z|^N \sum_{n=1}^{\infty} \frac{|z|^n}{(N-1)+n} \leqslant |z|^N \int_0^{\infty} \frac{|z|^t}{(N-1)+t}dt \qquad [45]$$

which, subject to the substitutions

$$|z| = e^{-p} \quad p > 0 \qquad [46a]$$

$$(N-1) = \lambda \qquad [46b]$$

becomes

$$|T_N(z)| \leq e^{-Np} \int_0^\infty \frac{e^{-pt}}{\lambda + t}\, dt \qquad \begin{matrix} |z| < 1 \\ p > 0 \end{matrix} . \tag{47}$$

Hence,

$$|T_N(z)| \leq e^{-p} \left| \mathcal{E}i\,(-\lambda p) \right| \tag{48}$$

where

$$\mathcal{E}i\,(\omega) = \int_{-\infty}^{\omega} \frac{\ell^t}{t}\, dt \qquad . \tag{48'}$$

Thus, the relative error $\mathcal{R}_N(z)$ in the sum is, for $z \geqslant 0$, bounded by

$$|\mathcal{R}_N(z)| \leq \frac{5}{2} \,|\, \mathcal{E}i\,(-\lambda p)\,| \qquad . \tag{49}$$

Taking $|\mathcal{R}_N(z)| < 5 \times 10^{-8}$ ($\lambda p \doteq 15$), one sees that $N = 64$ will suffice for $z \leqslant 0.75$. Hence, for use on digital machines it is suggested that the formula

$$\mathcal{B}_\gamma^{(1)}(z) = z^\gamma \sum_{n=o}^{64} \frac{z^n}{n+\gamma} \qquad \begin{matrix} 0 < z \leqslant 3/4 \\ -\frac{1}{2} \leqslant \gamma \leqslant 3/2 \\ \gamma \neq 0 \end{matrix} \tag{50}$$

be employed when possible.

Rather than use the methods outlined above to produce extensive tables of the $\mathcal{B}_\gamma^{(\ell)}(z)$, which would, in this era of digital machinery, be obsolescent from birth, the equations [39], [40], [42], and [50] were combined to produce a single subprogram for the IBM 7090 computer. Copies of this subprogram can be obtained by writing the author; a complete description of it is given in Appendix II.

## 4. Conclusions

It was shown that the problem of integrating the differential equation governing the uniform flow of liquid in an open channel can be reduced to the evaluation of three transcendental functions. This latter problem was shown to reduce to that of computing certain simple algebraic quantities and calculating various low-order polylogarithm functions.

No tables of backwater integrals were presented, since it was concluded that - in view of the extensive and rapidly growing use of digital computers - such tables would not be of widespread interest. Instead, a subprogram was written for use on an IBM 7090. With the aid of this subprogram the engineer can rapidly construct a program to calculate whatever flow profile he desires.

An area where considerable future progress can be made is that of the derivation of approximating polynomials for the several polylogarithm functions. A not unrealizable goal should be the polynomial approximation of $Li_2(z)$ which would reduce the evaluation of $\mathfrak{B}_{\gamma}^{(1)}(z)$ to the summation of a rather long series and the computation of the logarithm.

## APPENDIX I

### Construction of the Polynomial Approximations

The problem to be dealt with here is that of properly approximating certain transcendental functions. It was solved as follows.

Let f (x) be a function which is to be approximated by a polynomial $P_J(x)$ over an interval (0, 1). Let

$$D_J(x) = f(x) - P_J(x) \qquad . \tag{A-1}$$

Further, let $g(x)$ be some positive definite function over (0, 1), and let

$$Z_J(x) = D_J(x)/g(x) \tag{A-2}$$

be the error relative to the normalizing function $g(x)$. The problem is to adjust the coefficients $a_j$ of

$$P_J(x) = \sum_{j=0}^{J} a_j x^j \tag{A-3}$$

so that $Z_J(x)$ falls within some desired bounds.

The method used is based upon that of Spitzbart and Macon (4). Let $Z_J(x)$ be specified over a set $\{x_p\}$ of $J+2$ distinct points as

$$Z_J(x_p) = \lambda_p d \tag{A-4}$$

where the $\lambda_p$ have been specified and $d$ is some constant to be determined. Then

$$f(x_p) - P_J(x_p) = d\lambda_p g(x_p) \qquad . \tag{A-5}$$

d , a measure of the goodness of fit, and the coefficients $a_j$ are determined by solving the system [A-5] . Let

$$A = \begin{pmatrix} a_o \\ a_1 \\ \vdots \\ a_J \\ d \end{pmatrix} \qquad \text{[A-6a]}$$

$$F = \begin{pmatrix} f(x_o) \\ \vdots \\ f(x_{J+1}) \end{pmatrix} \qquad \text{[A-6b]}$$

$$W = \begin{pmatrix} 1 & x_o & \cdots & x_o^J & g(x_o) & \lambda_o \\ 1 & x_1 & \cdots & x_1^J & g(x_1) & \lambda_1 \\ \vdots & \vdots & & \vdots & \vdots & \\ 1 & x_{J+1} & & x_{J+1}^J & g(x_{J+1}) & \lambda_{J+1} \end{pmatrix} \qquad \text{[A-6c]}$$

With this abbreviated notation [A-5] can be rewritten as

$$WA = F \quad . \qquad \text{[A-7]}$$

[A-7] will have a non-trivial solution, i.e., $W^{-1}$ will exist, if and only if no polynomial of degree less than J+1 passes through the points

$$(x_p, \lambda_p\, g(x_p)) \qquad (p = 0, 1, 2, \ldots, J+1) \; .$$

For the approximations desired here this condition was met by setting

$$\lambda_p = (-1)^p \qquad (p = 0, 1, \ldots, J) \qquad \text{[A-8a]}$$

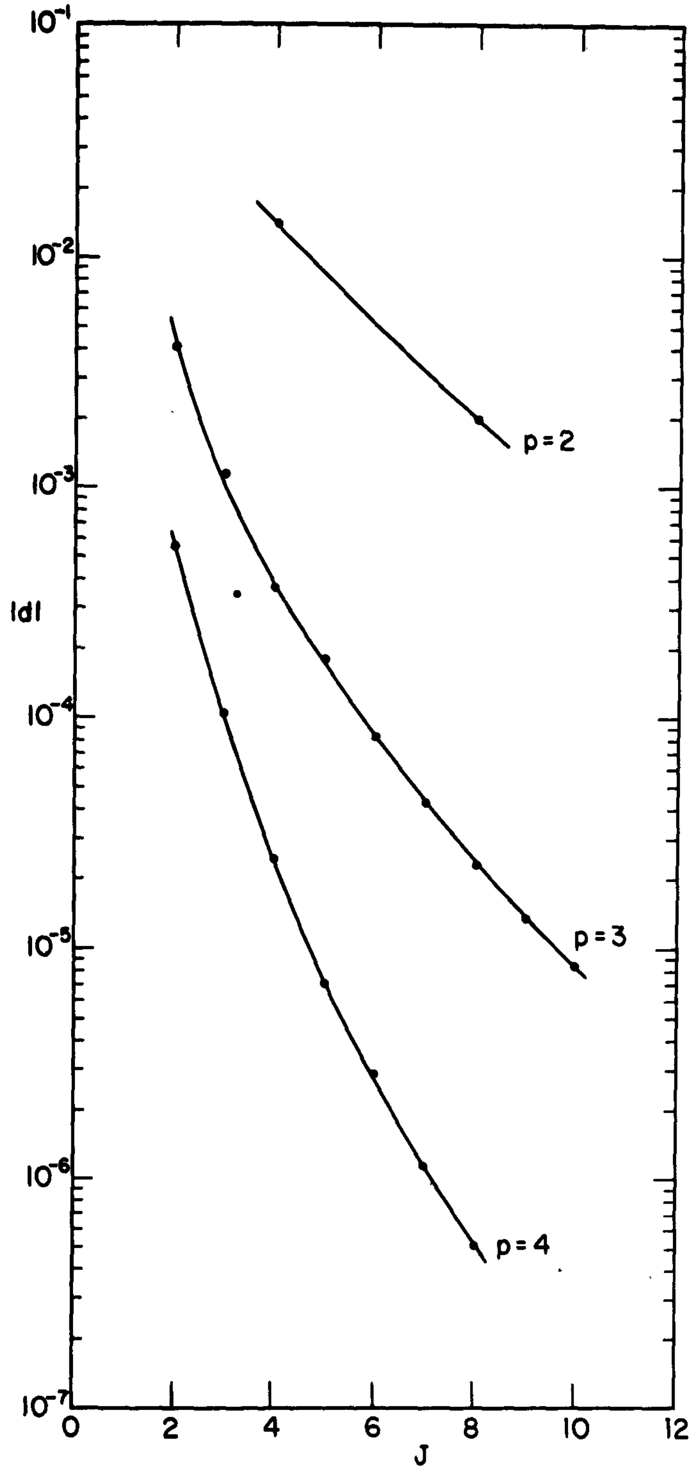


FIG. I  POLYNOMIAL APPROXIMATION OF $Li_p(z)/z$. ERROR CRITERION $|d|$ AS A FUNCTION OF p AND THE ORDER OF APPROXIMATION J.

and

$$\lambda_{J+1} = \lambda_J \ . \qquad [A-8b]$$

Of course, the formalism leading to [A-7] does not really solve the approximation problem completely, since it does not specify either $g(x)$ or the set $\{x_p\}$. The choice of $g(x)$ was left arbitrary and was varied from one $f(x)$ to another. The $\{x_p\}$ problem was solved by requiring the extremals of $Z_J(x)$ to coincide with $\{x_p\}$. For monotone $f(x)$ this leads to the danger of having only $J + 1$ extremals, and in this case $x_{J+1}$ was chosen to be 1, $x_J$ being unchanged if $x = 1$ was an extremal, or being placed between $x = x_{J-1}$ and $x = 1$ if $x = 1$ was an extremal. This choice of $\{x_p\}$ meant in practice that the set $\{x_p\}$ was found iteratively: a set $\{x_p^{(0)}\}$ was chosen, the $a_j$ found, and a curve of $Z_N(x)$ computed; from this $Z_N$ a new set of extremals, i.e., $\{x_p^{(1)}\}$, was found and the process repeated; the iteration was continued in this manner until convergence resulted or until $Z_N(x)$ and/or $d$ became so small that computing error rendered further iteration unprofitable. This method of selecting the $\lambda_p$ and the $\{x_p\}$ in effect adjusted $D_N(x)$ so that its extremals were of alternating sign and so that $|g(x)d|$ was an envelop for it.

A FORTRAN program embodying these features was written and run on the IBM 7090 computer at the Harvard University Computing Center. It was found that the W-matrices tended to be fairly poorly conditioned with the result that it was difficult to procure solutions for [A-7] when $J$ exceeded ten.

Since $\mathcal{B}_{\gamma}^{(1)}(z)$ is singular at $z = 1$, the weighting function

$$g(x) = 1 - \ln(1-z) \qquad [A\text{-}9]$$

was chosen, and approximations to $Li_p(z)/z$ for various J and $p = 2, 3, 4$ were generated. The results of these calculations are summarized in Fig. I. The singularity at $z = 1$ in the $p^{th}$ derivative of $Li_p(z)$ is seen to affect powerfully the goodness of fit. The case $p = 4$, $J = 8$ was chosen as adequate and the results presented in [41] were adopted. With an approximating program of higher accuracy it seems likely that sufficiently accurate results could be obtained with $p = 3$, $J > 16$. It seems probable that the obtaining of a good approximation for $p = 2$ will require a quite large value of J.

The fitting of $\mathcal{B}_{\gamma}^{(3)}(z)$ was much simpler, since it and its derivatives possess no singularity over (0, 1). In this case a weight function equal approximately to unity was chosen and the results quoted in [38] obtained for $p = 1$, $J = 8$.

## APPENDIX II

### A Program for Computing the Backwater Integrals

A FAP coded FORTRAN type subprogram, named BKW, was written to compute the backwater integrals. It was subsequently debugged and tested on the IBM 7090 at the Harvard University Computing Center. Binary decks for this routine may be obtained by writing the author; these decks also contain the special subroutines AGLOG, AGPWR, and AGTLG called by BKW.

A listing of BKW is presented in Figs. IIa through IIe. The program for computing $\mathcal{B}_\gamma^{(\ell)}(z)$ is seen to be divided roughly into five major subsections. In the first subsection the program decides which of the three integrals ($\ell = 1, 2, 3$) is to be evaluated and transfers control to the appropriate subsection. There is a separate subsection (B1, B2, or B3) for each value of $\ell$. Finally, there is a subsection (ERR and/or RET) which controls returns to the calling program.

B1. This subsection first examines z: for $z \leqslant 0$ or $z \geqslant 1$, an error return (transfer to ERR) is specified, for $0 < z \leqslant 0.75$, an evaluation by the series [50] (transfer to SER1) is specified; for $0.75 < z < 1.00$, an evaluation by means of Eq. [42] is specified. The first step in the evaluation by either [50] or [42] is the examination of $\gamma$: if the $\gamma$ specified equals a value held in storage (zero if this is the first pass through SER1 or the corresponding routine for large z, otherwise the value of $\gamma$ on the last pass through the appropriate routine), the values of $\frac{1}{n+\gamma}$ (or ${}_4^9 c_j^{(+)} \frac{j}{j+\gamma}$ ) are known to be computed and held in storage already so that $\mathcal{B}_\gamma^{(1)}(z)$ can be computed at once from [42] (or [50] ) and control transferred to RET; if $\gamma$ does not agree with the value held in storage, it is checked to make sure that it is within the prescribed range of values and, assuming that it passes this test (transfer to ERR otherwise), the appropriate constants are calculated and the integral evaluated as before. In the course of these calculations various special subroutines are utilized: AGLOG, a 27 bit accuracy, floating point logarithm routine; AGPWR, a 26 bit accuracy, floating point power routine; AGDLG, a routine for $Li_2(z)$; AGTLG, a routine for $Li_3$ (z).

B2. This subsection first obtains $\gamma$ and z and tests to see that they lie within the allowed ranges. Failure of this test causes program control to be transferred to ERR. Success of this test results in the evaluation of $\mathcal{B}_\gamma^{(2)}(z)$ directly from [40]; control is then transferred to RET.

B3. This subsection is roughly similar to that for $\mathcal{B}_\gamma^{(1)}(z)$, except that the integrals for all allowable z are computed from the same formula. First, z is checked for range. Then $\gamma$ is checked, and, if necessary, the constants ${}_1^9 c_j^{(-)} \frac{j}{j+\gamma}$ are recomputed. Finally, $\mathcal{B}_\gamma^{(3)}$ (z) is computed from [39], and control is transferred to RET.

ERR and RET. ERR is an error routine which clears the accumulator and turns on sense light 4; it then transfers control to RET. RET resets the three index registers and returns control to the calling program.

* * *

USAGE. The appropriate calling sequence in FAP is

```
A     TSX     BKW, 4
      TSX     L
      TSX     GAMMA
      TSX     Z
```

The normal return is to A+4 within $\mathcal{B}_{\gamma}^{(\ell)}(z)$ in the accumulator; in case of an error in the range of z or $\gamma$, return is to A+4 but the accumulator will be clear and sense light 4 will be on. (Note that L must be a FORTRAN integer and that GAMMA and Z must be FORTRAN floating point numbers.) The appropriate calling sequence in FORTRAN is

Y = BKW(L, GAMMA, Z)

where Y is the value of the desired integral.

TIMING. This is quite variable. It runs from about 10 ms for the worst cases of $\mathcal{B}_{\gamma}^{(1)}(z)$ down to $2^{+}$ ms for $\mathcal{B}_{\gamma}^{(3)}(z)$ and 2 ms for $\mathcal{B}_{\gamma}^{(2)}(z)$.

ACCURACY. Error is expected to be less than 1 part in $10^5$ in most cases. For $\mathcal{B}_{\gamma}^{(1)}(z)$ $(0.75 < z < 1.00)$ it is normally less than 1 part in $10^6$.

```
        ENTRY   BKW
BKW     SXD     *-2,4
        SXD     ERAS+1,1          SAVE INDICES
        SXD     ERAS+2,2
        SXD     ERAS+4,4
        CLA*    1,4               GET L
        CAS     CNST
        TRA     *+2
        TRA     B1                L = 1
        CAS     CNST+1
        TRA     *+2
        TRA     B2                L = 2
        CAS     CNST+2
        TRA     *+2
        TRA     B3                L = 3
        TRA     ERR
B1      CLA*    2,4               GET AND SAVE GAMMA
        STO     ERAS
        CLA*    3,4               GET Z
        TMI     ERR               Z NEGATIVE
        TZE     ERR               Z ZERO
        CAS     CNST+3
        TRA     ERR               Z TOO LARGE
        TTR     *-1
        STO     Z                 Z OK.  SAVE
        CAS     CNST+4
        TRA     *+6               Z GT 0.75
        TTR     *+1
        TSX     SER1,4
        HTR     ERAS
        HTR     Z
        TRA     RET
        CLA     ERAS
        CAS     G1
        TTR     *+2
        TRA     *+2               OLD CONSTANTS SUFFICE
        TRA     CON1              GET NEW CONSTANTS FOR ASYMP EVAL
        CLA     CNST+3            BEGIN ASYMPTOTIC EVALUATION
        FDP     G1
        STQ     TOT
        CLA     CNST+3
        FSB     Z
        STO     ERAS
        TSX     $AGLOG,4
        HTR     FRAS
        HTR     ERAS
        CHS
        FAD     TOT
        STO     TOT               LI1 INCLUDED
        TSX     $AGDLG,4
        HTR     Z
        XCA
        FMP     G1
        CHS
        FAD     TOT
        STO     TOT               LI2 INCLUDED
        TSX     $AGTLG,4
        HTR     Z
```

**FIG. IIa LISTING OF BKW (PAGE 1 OF 5)**

```
         XCA
         FMP     G1
         XCA
         FMP     G1
         FAD     TOT
         STO     TOT              LI3 INCLUDED
         AXT     0,1              START APPROXIMATING SERIES
         PXD     0,0
         FAD     S1+8,1
         XCA
         FMP     Z
         TXI     *+1,1,1
         TXL     *-4,1,8
         XCA
         FMP     G1
         XCA
         FMP     G1
         XCA
         FMP     G1
         CHS
         FAD     TOT
         STO     TOT              COMPLETE SERIES
         TSX     $AGPWR,4
         HTR     Z
         HTR     G1
         XCA
         FMP     TOT              B1 COMPLETE
         TRA     RET
SER1     SXD     ERAS+3,4         SAVE XR(4)
         CLA*    2,4
         STO     Z                SAVE Z
         CLA*    1,4              GET GAMMA
         CAS     G1S
         TTR     *+2
         TRA     *+2              OLD CONSTANTS SUFFICE
         TRA     CON1S            NEW CONSTANTS NECESSARY
         AXT     0,4              BEGIN SERIES SUMMATION
         PXD     0,0
         FAD     S1S+64,4
         XCA
         FMP     Z
         TXI     *+1,4,1
         TXL     *-4,4,63
         FAD     S1S
         STO     ERAS
         TSX     $AGPWR,4
         HTR     Z
         HTR     G1S
         XCA
         FMP     ERAS
         LXD     ERAS+3,4
         TRA     3,4
CON1     TZE     ERR              CHECK RANGE OF GAMMA
         CAS     CNST+5
         TRA     *+3
         TTR     *-1
         TRA     ERR
         CAS     CNST+6
```

FIG. IIb LISTING OF BKW (PAGE 2 OF 5)

```
        TRA    ERR
        TTR    *+1
        STO    G1                SAVE NEW VALUE OF GAMMA
        AXT    0,1               SET UP NEW CONSTANTS
        CLA    CNST+7
        STO    ERAS+5
        CLA    ERAS+5
        FAD    G1
        STO    ERAS+6
        CLA    ERAS+5
        FDP    ERAS+6
        FMP    C+8,1
        STO    S1+8,1
        CLA    ERAS+5
        FSB    CNST+3
        STO    ERAS+5
        TXI    *+1,1,1
        TXL    *-11,1,8
        TRA    B1+21             RETURN TO B1 EVALUATION
CON1S   TZE    ERR               CHECK RANGE OF GAMMA
        CAS    CNST+5
        TRA    *+3
        TTR    *-1
        TRA    ERR
        CAS    CNST+6
        TRA    ERR
        TTR    *+1
        STO    G1S               SAVE NEW VALUE OF GAMMA
        AXT    0,4               SET UP NEW CONSTANTS
        CLA    CNST+8
        STO    ERAS+5
        CLA    ERAS+5
        FAD    G1S
        STO    ERAS+6
        CLA    CNST+3
        FDP    ERAS+6
        STQ    S1S+64,4
        CLA    ERAS+5
        FSB    CNST+3
        STO    ERAS+5
        TXI    *+1,4,1
        TXL    *-10,4,64
        TRA    SER1+8            RETURN TO SER1
ERR     PSE    100               TURN ON SENSE LIGHT 4
        PXD    0,0               CLEAR AC
        TRA    RET               PREPARE TO EXIT
RET     LXD    ERAS+1,1
        LXD    ERAS+2,2
        LXD    ERAS+4,4
        TRA    4,4               GO BACK TO MAIN PROGRAM
B2      CLA*   2,4               OBTAIN GAMMA AND CHECK RANGE
        STO    G2
        TMI    ERR
        TZE    ERR
        CAS    CNST+9
        TRA    ERR
        TTR    *+1
        CLA*   3,4               OBTAIN Z AND CHECK RANGE
```

FIG. IIc LISTING OF BKW (PAGE 3 OF 5)

```
        TMI    ERR
        TZE    ERR
        STO    Z
        TSX    $AGPWR,4            Z**GAMMA
        HTR    Z
        HTR    G2
        FDP    G2
        XCA
        TRA    RET
B3      CLA*   2,4                 GET GAMMA
        CAS    G3
        TRA    *+2
        TRA    *+2                 OLD CONSTANTS SUFFICE
        TRA    CON3                NEW CONSTANTS NEEDED
        CLA*   3,4                 GET AND TEST Z
        TMI    ERR
        TZE    ERR
        CAS    CNST+3
        TRA    ERR
        TTR    *+1
        STO    Z
        AXT    0,1                 SUM SERIES
        PXD    0,0
        FAD    S3+8,1
        XCA
        FMP    Z
        TXI    *+1,1,1
        TXL    *-4,1,8
        STO    ERAS
        CLA    CNST+3
        FDP    G3
        XCA
        FAD    ERAS
        STO    ERAS                FIND B3
        TSX    $AGPWR,4
        HTR    Z
        HTR    G3
        XCA
        FMP    ERAS
        TRA    RET
CON3    TZE    ERR                 TEST GAMMA
        CAS    CNST+5
        TRA    *+3
        TTR    *-1
        TRA    ERR
        CAS    CNST+6
        TRA    ERR
        TTR    *+1
        STO    G3                  GAMMA OK
        AXT    0,2                 SET UP SERIES REVISION
        CLA    CNST+7
        STO    ERAS+6
        CLA    ERAS+6              REVISE
        FAD    G3
        STO    ERAS
        CLA    D+8,2
        FDP    ERAS
        FMP    ERAS+6
```

**FIG. IId LISTING OF BKW (PAGE 4 OF 5)**

```
        STO     S3+8,2
        CLA     ERAS+6
        FSB     CNST+3
        STO     ERAS+6
        TXI     *+1,2,1
        TXL     *-11,2,8
        TRA     S3+5
CNST    PZE     0,0,1
        PZE     0,0,2
        PZE     0,0,3
        DEC     1.00
        DEC     0.75
        DEC     -0.50
        DEC     1.50
        DEC     9.00
        DEC     64.00
        DEC     7.00
C       DEC     +0.100000046E+01      LI4 CONSTANTS
        DEC     +0.624179326E-01
        DEC     +0.143776655E-01
        DEC     -0.150221251E-01
        DEC     +0.873512559E-01
        DEC     -0.208788007E-00
        DEC     +0.283126429E-00
        DEC     -0.198408261E-00
        DEC     +0.572706185E-01
D       DEC     -0.999999993E+00      LI1 CONSTANTS
        DEC     +0.499989465E-00
        DEC     -0.333077177E-00
        DEC     +0.247635670E-00
        DEC     -0.188352078E-00
        DEC     +0.131561905E-00
        DEC     -0.729025990E-01
        DEC     +0.265650421E-01
        DEC     -0.456741825E-02
ERAS    BSS     7
Z       BSS     1
G1      BSS     1
G1S     BSS     1
G2      BSS     1
G3      BSS     1
TOT     BSS     1
S1      BSS     9
S1S     BSS     65
S3      BSS     9
        END
```

FIG. IIe LISTING OF BKW (PAGE 5 OF 5)

**For $\mathcal{B}_{\gamma}^{(3)}(z)$ it is normally less than 2 parts in $10^7$ and often less than 2 parts in $10^8$. For $\mathcal{B}_{\gamma}^{(1)}(z)$ $(0.00 < z \leqslant 0.75)$ the error is normally confined to the eighth figure. The error in $\mathcal{B}_{\gamma}^{(2)}(z)$ is normally less than 2 parts in $10^8$. Regions of high relative error can be expected in $\mathcal{B}_{\gamma}^{(1)}(z)$ for $\gamma < 0$ and values of $z$ for which $\mathcal{B}_{\gamma}^{(1)}(z) \doteq 0$ ; however, even here several good decimal places will normally be available.**

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